Incommensurate Charge Density Waves in the adiabatic Hubbard-Holstein model

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The adiabatic, Holstein-Hubbard model describes electrons on a chain with step a interacting with themselves (with coupling U) and with a classical phonon field φ_x (with coupling λ). There is Peierls instability if the electronic ground state energy $F(\varphi)$ as a functional of φ_x has a minimum which corresponds to a periodic function with period $\frac{\pi}{p_F}$, where p_F is the Fermi momentum. We consider $\frac{p_F}{\pi a}$ irrational so that the CDW is *incommensurate* with the chain. We prove in a rigorous way in the spinless case, when λ, U are small and $\frac{U}{\lambda}$ large, that a)when the electronic interaction is attractive $U < 0$ there is no Peierls instability b)when the interaction is repulsive $U > 0$ there is Peierls instability in the sense that our convergent expansion for $F(\varphi)$, truncated at the second order, has a minimum which corresponds to an analytical and $\frac{\pi}{p_F}$ periodic φ_x. Such a minimum is found solving an infinite set of coupled self-consistent equations, one for each of the infinite Fourier modes of φ_x.

I. INTRODUCTION

In 1955 Peierls, in Ref. [27], suggested that in a one dimensional metal it is energetically favorable to develop a periodic distortion of the linear lattice with period $\frac{p_F}{\pi}$ where p_F is the Fermi momentum of the conduction electrons. The attempt of the conduction electrons to screen the periodic potential generated by the periodic lattice distortion creates a *Charge Density Wave* (CDW) in the conduction electron density. If a is the step of the undistorted lattice, then, depending whether $\frac{p_F a}{\pi}$ is a rational number or not, the CDW (or the periodic lattice distortion) can be either *commensurate* or *incommensurate* with the non distorted lattice. While a commensurate CDW has preferred positions in the lattice, an incommensurate CDW has not and so it can slide without any change of energy; this was considered by Frolich in Ref. [13] a possible mechanism for superconductivity or a least for an enhancement of conductivity (see Ref. [23]).

Indeed starting from the 70's both commensurate or incommensurate CDW with wavevector $2p_F$ have indeed been observed in a number of compounds (see for instance Refs. [30] and [18]). A new wind of interest followed recently the discovery of high T_c superconductors showing one dimensional incommensurate CDW, see Refs. [4] and [28].

From a theoretical point of view, Peierls instability can be studied in the *Holstein-Hubbard* model, which is the simplest model involving both an electron-phonon and an electron-electron interaction. The standard theory of CDW is usually developed within the adiabatic approximation, in which the phonon field is treated as a classical field and the model becomes variational. We will consider the spinless case, so that the *spinless adiabatic Holstein-Hubbard* hamiltonian is

$$H = H_0 + H_p + \lambda P + UV = \tag{1}$$
$$\sum_{x,y\in\Lambda} (t_{xy} - \mu\delta_{x,y})\, \psi^+_x \psi^-_y + \frac{1}{2}\sum_{x\in\Lambda} \varphi_x^2 - \lambda \sum_{x\in\Lambda} \varphi_x \psi^+_x \psi^-_x + U \sum_{x\in\Lambda} [\psi^+_x \psi^-_x - \frac{1}{2}][\psi^+_{x+1}\psi^-_{x+1} - \frac{1}{2}],$$

where $\Lambda = 1, ..., L-1$, $t_{x,y} = \delta_{x,y} - \frac{1}{2}(\delta_{x,y+1} + \delta_{x,y-1})$, $\psi^\pm_x$ are fermionic creation or annihilation fields with periodic boundary conditions, μ is the chemical potential $\mu = 1 - \cos p_F$. In (1) H_0 is the fermionic kinetic energy, H_p is the phonon kinetic energy, λP is the electron-phonon interaction and UV is the electron-electron interaction; to describe the Coulomb repulsion one needs $U > 0$ but it is not irrealistic to consider also $U < 0$ (in this case UV is an effective interaction taking into account phonon-mediated processes). If $U = 0$ the above model is called *Holstein model*. p_F is the Fermi momentum of the non interacting $U = 0$ model; the Fermi momentum in the $U \neq 0$ case is in general different, but we fix it to p_F by adding a term νN to the hamiltonian, where $N = \sum_{x\in\Lambda} \psi^+_x \psi^-_x$ is the total particle number operator and ν is a suitable counterterm. The proof of Peierls instability consists, within this model, in the proof that the ground state energy $F(\varphi) = \sum_{x\in\Lambda} \frac{\varphi_x^2}{2} + E_0(\varphi)$ is minimized by $\varphi_x = \bar\varphi(2p_F x)$ where $\bar\varphi(t)$ is a 2π-periodic function. The existence of a global minimum of the form $\bar\varphi(2p_F x)$ was proved in the half filled band case $p_F = \pi/2$ in the Holstein model and in the spinning Holstein-Hubbard model (see Refs. [20] and [21]). Local minima of the form $\bar\varphi(2p_F x)$, for any $p_F = \pi\frac{P}{Q}$ with P, Q relatively prime, were found in Ref. [6], for $|\lambda| \leq O(\log Q^{-1})$. We are interested here in the *incommensurate* case in which $\frac{p_F}{\pi}$ is an irrational number. At finite L, it is not possible to fix $\frac{p_F}{\pi}$ directly to an irrational number, as in this way $\bar\varphi(2p_F x)$ cannot verify periodic boundary conditions. We

look however to a sequence of L_i, n_i such that $\lim_{i\to\infty} L_i = \infty$ and $\lim_{i\to\infty} p_{F,i} = p_F$ and $\frac{p_F}{\pi}$ is irrational, where $p_{F,i} = \frac{2\pi n_i}{L_i}$. An *incommensurate* phonon field has the form

$$\varphi_x = \lim_{i\to\infty} \sum_{n=-[L_i/2]}^{[(L_i-1)/2]} \hat{\varphi}_n e^{i2p_{F,i}nx}. \tag{2}$$

We require moreover that

$$\|2np_{F,i}\|_{\mathbf{T^1}} \geq C_0|n|^{-\tau}\ , \quad 0 \neq n \in Z, \qquad |n| \leq \frac{L_i}{2}, \tag{3}$$

where $||k||_{\mathbf{T^1}} = \min_{n\in Z} |k - 2\pi n|$. (3) means that p_F/π is a *Diophantine number* (see for instance Ref. [15]) and the proof of the existence of $p_{F,i}$ verifying (3) can be found in Ref. [5]. Such assumption is not really restrictive, as if $\tau > 1$ the complementary set of such points has measure 0. A Diophantine condition like (3) (in the infinite volume limit) appears in classical mechanics, for instance in the KAM theorem, see Refs. [1] and [26], and it is useful for handling with the so called *small divisor problem*; a similar problem appears also here. That a sort of extension of KAM-techniques to quantum systems is necessary to prove Peierls instability in the incommensurate case was pointed out for the first time by Aubry in Ref. [3] by analogy with the *Frenkel-Kontorova* models (see also Ref. [2]). We have to specify the space of functions on which $F(\varphi)$ is defined as a variational form. We say that $F(\varphi)$ is a functional $F : \Omega \to R$ where Ω *is the set of functions* φ_x *of the form (2) with zero average* $\hat{\varphi}_0 = 0$ *and* $\hat{\varphi}_n = \hat{\varphi}_{-n} = \hat{\varphi}_n^*$. *Moreover, if* $\kappa, F_0, F_1 > 0$ *are constants and* $\sigma = \lambda\hat{\varphi}_1$ *then* $|\sigma| \leq F_0$ *and, for* $|n| > 1$

$$|\lambda\hat{\varphi}_n| \leq F_1|\sigma|e^{-\kappa|n|}. \tag{4}$$

The condition (4) ensures also that the $2p_F$ harmonic is present (if $\hat{\varphi}_1 = 0$ then φ_x is a constant). If φ is an extremal point of $F(\varphi)$, it must satisfy the condition $\hat{\varphi}_0 = \lambda\rho$ where ρ is the fermionic density. On the other hand, we can always include $\hat{\varphi}_0$ in the chemical potential μ and then we can restrict our search of local minima of the ground state energy $F(\varphi)$ to fields φ with zero mean. Note that $F : \Omega \to R$ is indeed a function of the Fourier coefficients $\hat{\varphi}_n$; this means that, at finite L, can be considered not a functional but a L-dimensional function and only at the end we will take the $L \to \infty$ limit.

We say that *there is Peierls instability if the variational form* $F : \Omega \to R$ *has a minimum* $\varphi_x \in \Omega$. We will show in §2 that if $\varphi_x \in \Omega$ is a local minimum then

$$\hat{\varphi}_n = \lambda\hat{\rho}_n(\varphi) \quad n \neq 0, \quad n = -[L/2], ..., [(L-1)/2], \tag{5}$$

and $M_{nm} = \delta_{nm} - \lambda\frac{\partial}{\partial\hat{\varphi}_n}\hat{\rho}_m(\varphi)$ positive definite, where $\hat{\rho}_n(\varphi) = \frac{\partial E_0(\varphi)}{\partial\hat{\varphi}_n}$. Peierls instability can be proved by solving the infinite (as $L \to \infty$) set of coupled equations (5). There are then two main steps to be performed; the first is to compute $\hat{\rho}_n$ by an expansion, as there is no hope to compute it in a simple exact form, and the second is to solve the system $\hat{\varphi}_n = \lambda\hat{\rho}_n(\varphi)$. In §2 an expansion for $\hat{\rho}_n(\varphi)$ is found, which is *convergent* for any $\varphi \in \Omega$, for λ, U small enough. The proof of convergence is based on a sort of generalization of KAM theorem to quantum system; in §2 we review the main ideas referring to Ref. [24] for the mathematical proofs. The result is that we can write $\hat{\rho}_n = \sum_{k=0}^{\infty}\hat{\rho}_n^{(k)}$ and, *if* $p_{F,i}$ *verifies (3) and* $\varphi_x \in \Omega$, *then for* $|\lambda|, |U| \leq \varepsilon$ *one has* $|\hat{\rho}_n^{(k)}| \leq f(n,\lambda,U)C^k\varepsilon^k$, *where* C *is a constant and* $f(n,\lambda,U)$ *is a proper function.* Of course the exact form of $f(n,\lambda,U)$ is important, and it will be specified in §2.E. The proof of the convergence (for ε small enough) of the expansion for the density is based on Renormalization Group methods; it is important however to stress that, while the usual RG methods are only approximative, our results are *mathematically exact*; we refer for an introduction to rigorous RG for fermions to Ref. [16].

The second step (see §3) consists in solving (5). This is quite a difficult task, and we consider the simpler problem obtained by keeping only the first two terms of the expansion for $\hat{\rho}_n$. This means that we study

$$\hat{\varphi}_n = \lambda\hat{\rho}_n^{(0)}(\varphi) + \lambda\hat{\rho}_n^{(1)}(\varphi), \qquad n = -[L/2], ..., [(L-1)/2]. \tag{6}$$

so neglecting $O(\varepsilon^2)$ terms in the r.h.s. of (5). The convergence of the expansion for $\hat{\rho}_n$ makes this approximation quite reasonable. Note also that $\hat{\rho}_n^{(0)}$ and $\hat{\rho}_n^{(1)}$ are rather complex function of λ, U (the expansion is not a power series in λ, U). Our main result is that *if* $\frac{|\lambda|}{|U|}$ *is small enough, in the* $L \to \infty$ *limit*

a)in the attractive $U < 0$ *case there are no solutions* $\varphi \in \Omega$ *of (6) (and of (5) as well);*

b)in the repulsive case $U > 0$ *there is a* $\varphi_x \in \Omega$ *solving (6) such that*

$$\lambda\hat{\varphi}_1 \equiv \sigma = A[\frac{\lambda^2}{a\eta}]^{\frac{1}{\eta}}[1+O(\frac{\lambda^2}{U})]^{\frac{1}{\eta}}; \qquad |\hat{\varphi}_n| \le e^{-\frac{|\log|\lambda||}{10}|n|}|\sigma| \qquad |n| \neq 1. \tag{7}$$

with $\eta = \beta_1 U + O(U^2)$ *is a critical index and* a, A *positive constants.* This means that, if the electron-electron interaction is larger than the electron-phonon interaction, there is a dramatic dependence on the repulsive or attractive nature of the electron-electron interaction. In the attractive case there is no Peierls instability as there are no functions analytical and $\frac{\pi}{p_F}$ periodic minimizing the ground state energy. In the repulsive case there is instead Peierls instability, in the sense that the energy, keeping the first two non trivial terms of its convergent expansion, has a minimum which corresponds to an incommensurate CDW. It is very reasonable that higher order terms do not change this result (see the considerations in §3.B).

Our results improve preceding works on the subject in which a)the interaction among electrons was neglected; b)the analysis was restricted to the *first order*

$$\hat{\varphi}_n = \lambda\hat{\rho}_n^{(0)}(\varphi), \tag{8}$$

where, if $U = 0$, $\lambda\hat{\rho}_n^{(0)}(\varphi) \simeq -a\lambda^2\hat{\varphi}_1 \log|\lambda\hat{\varphi}_1|$ if $n = \pm 1$ and zero otherwise. There is a major difference between (8) and (6); while (8) admits trivially a solution $\lambda\hat{\varphi}_n = \sigma\delta_{n,\pm 1}$ and σ is obtained by a BCS like equation, a solution of the form $\lambda\hat{\varphi}_n = \sigma\delta_{n,\pm 1}$ *does not solve* (6) or (5). While a solution of (8) if $U = 0$ (when the model reduces to the Holstein model) is trivial to find, we are not able to find a solution of (6) $\in \Omega$ if $U = 0$, as the method we use to find a solution of (6) when $U > 0$ and $\frac{U}{|\lambda|}$ is large fails in that case. We have then no evidence of Peierls instability in the Holstein model and it is unclear if this is merely a technical problem or if there is $U_c > 0$ such that Peierls instability holds only for $U > U_c$ in the Holstein-Hubbard model.

The paper is organized in the following way. In §II.A we describe an expansion for the ground state energy of the Hubbard-Holstein model (1) for $\varphi \in \Omega$. In order to prove the convergence (for small λ, U) of the expansion for small λ, U, one has to solve a small divisor problem, and this is discussed in §II.B. In §II.C it is studied the RG flow and in §II.D we prove that the renormalization of the Fermi momentum if λ-independent. From the ground state energy expansion, is it easy to derive an expansion for the density, and this is done in §II.E. In §III we prove that (5) has no solution $\varphi \in \Omega$ in the attractive case, if $\frac{|\lambda|}{|U|}$ is small enough, while in the repulsive case we find a solution $\varphi \in \Omega$ of (6) by a contraction method. Finally in §IV we discuss some open problems, in particular for the $U = 0$ case.

II. RENORMALIZATION GROUP ANALYSIS

A. Grassman integrals

It is well known that $E_0(\varphi)$ can be written as a *Grassman integral* (we use the same symbol ψ for field and Grassman variables with a traditional abuse of notation)

$$E_0(\varphi) = -\lim_{\beta\to\infty} \frac{1}{L\beta} \log \int P(d\psi) e^{-UV-\lambda P-\nu N}, \tag{9}$$

where

$$V = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dx_0 \sum_{x\in\Lambda} [\psi_{\vec{x}}^+\psi_{\vec{x}}^- - \frac{1}{2}][\psi_{\vec{x}+1}^+\psi_{\vec{x}+1}^- - \frac{1}{2}], \quad P = -\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dx_0 \sum_{x\in\Lambda} \varphi(x)\psi_{\vec{x}}^+\psi_{\vec{x}}^-, \quad N = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dx_0 \sum_{x\in\Lambda} \psi_{\vec{x}}^+\psi_{\vec{x}}^-, \tag{10}$$

and $\vec{x} = (x_0, x)$ and $\vec{x}+1 = (x_0, x+1)$. $P(d\psi)$ is a *Grassmanian integration* defined on monomials by the anticommutative Wick rule with propagator

$$g(\vec{x};\vec{y}) = \frac{1}{\beta L}\sum_{\vec{k}} \frac{e^{-i\vec{k}(\vec{x}-\vec{y})}}{-ik_0 - \cos k + \cos p_F}, \tag{11}$$

where $\vec{k} = (k_0, k)$. (9) has a well defined $L, \beta \to \infty$ limit only if the counterterm ν is chosen in a suitable way as a function of the parameters appearing in the Hamiltonian so that the Fermi momentum is just p_F. In order to find the minima of $F(\varphi)$ we have to differentiate with respect to $\hat{\varphi}_n$, so one has in principle to take into account the

possible dependence of ν from $\hat\varphi_n$, which is in general very complicated. However we will show in §2.D that it is possible to choose ν as *independent* from λ and so from $\hat\varphi_n$. This is due to the fact that the chemical potential can be moved inside the gap opened by φ_x without affecting any physical property, and we can use this freedom to fix ν as independent of $\hat\varphi_n$. It follows that a necessary condition for $\varphi_x \in \Omega$ to be a local minimum for $F(\varphi)$ is that it verifies $\varphi_x = \lambda\rho_x$ where $\rho_x = \lim_{\beta\to\infty,\tau\to 0} \frac{1}{L} S^{L,\beta}(x,\tau;x,0)$ and $S^{L,\beta}(x,\tau;x,0)$ is the Schwinger function defined by, if $\phi^\pm_{\vec x}$ are Grassman variables and writing $\int d\vec x = \int_{\frac{-\beta}{2}}^{\frac{\beta}{2}} \sum_{x\in\Lambda}$,

$$S^{L,\beta}(\vec x;\vec y) = \frac{\partial^2}{\phi^+_{\vec x}\partial\phi^-_{\vec y}} \log \int P(d\psi) e^{-\mathcal{V}(\psi) - \int d\vec x [\phi^+_{\vec x}\psi^-_{\vec x} + \phi^-_{\vec x}\psi^+_{\vec x}]}|_{\phi=0}, \tag{12}$$

and $\mathcal{V} = UV + \lambda P + \nu N$. (5) is obtained by the Fourier transform of $\varphi_x = \lambda\rho_x$ defining $\rho_x = \sum_{n=-[L/2]}^{[(L-1)/2]} e^{2inp_F x}\hat\rho_n$.

The above Grassman integrals can be evaluated by Renormalization Group methods; we refer to Ref. [16] for an introduction to the formalism we are using and to Ref. [24] for the mathematical proofs of the convergence of the expansion we are describing. We start by evaluating the partition function $\int P(d\psi) e^{-\mathcal{V}(\psi)}$. It is convenient to decompose the Grassman integration $P(d\psi)$ into a product of independent integrations. Let be $|\vec k| = \sqrt{k_0^2 + ||k||_T^2}$. We write

$$g(\vec k) = f_1(\vec k)g(\vec k) + (1 - f_1(\vec k))g(\vec k) = g^{(u.v.)}(\vec k) + g^{(i.r.)}(\vec k), \tag{13}$$

where $f_1(\vec k) = 1 - \chi(k - p_F, k_0) - \chi(k + p_F, k_0)$ and $\chi(k', k)$ is a C^∞ function with compact support such that it is 1 for $|\vec k'| \le \frac{a_0}{\gamma}$ and 0 for $|\vec k'| > a_0$, where $\gamma > 1$ and γ, a_0 are chosen so that $\chi(k \pm p_F, k_0)$ are non vanishing only in two non overlapping regions around $\pm p_F$. We write $k = k' + \omega p_F$, $\omega = \pm 1$ and

$$g^{(i.r.)}(\vec k) = \sum_{\omega=\pm1}\sum_{h=-\infty}^{0} f_h(\vec k')g(\vec k) \equiv \sum_{\omega=\pm1}\sum_{h=-\infty}^{0} g^{(h)}(\vec k), \tag{14}$$

where $f_h(\vec k') = \chi(\gamma^{-h}\vec k') - \chi(\gamma^{-h+1}\vec k')$ has support $O(\gamma^h)$ around ωp_F. The integration of $\psi^{(1)}$, the *ultraviolet* integration, gives

$$e^{-\mathcal{V}^{(0)}(\psi^{(\le 0)})} = \int P(d\psi^{(1)}) e^{-\mathcal{V}(\psi^{(1)} + \psi^{(\le 0)})}, \tag{15}$$

where $\psi^{\le 0} = \sum_{k=-\infty}^0 \psi^{(k)}$ and (denoting $\frac{1}{\beta L}\sum_{\vec k}$ simply by by $\int d\vec k$)

$$\mathcal{V}^{(0)}(\psi^{(\le 0)}) = \sum_{n=1}^\infty \sum_{m=0}^\infty \int d\vec k_1 ... d\vec k_{2n} \psi^{(\le 0)\sigma_1}_{\vec k_1} ... \psi^{(\le 0)\sigma_{2n}}_{\vec k_{2n}} W^0_{2n,m}(\vec k_1, ..., \vec k_{2n}) \delta(\sum_{i=1}^{2n} \sigma_i \vec k_i + 2m\vec p_F), \tag{16}$$

where $\vec p_F = (p_F, 0)$, $\sigma_i = \pm$ and the kernels $W^0_{n,m}(\vec k_1, ..., \vec k_n; z)$ are C^∞ bounded functions such that $W^0_{n,m} = W^0_{n,-m}$ and $|W^0_{n,m}| \le C^n z^{\max(2, n/2-1)}$ if $z = Max(|\lambda|, |U|, |\nu|)$. By an explicit computation it follows that $W^0_{4,0} = U + O(U^2)$, $W^0_{4,m} = O(U\sigma)$ for $m \ne 0$ and $W^0_{2,m} = \sigma + O(\sigma U)$ for $m \ne 0$. $\mathcal{V}^{(0)}$ is called *effective potential at scale* 0; note that it contains non local interactions between an arbitrary number of fermions.

The study of the *infrared* integration is much more involved. Following Wilsonian Renormalization group methods, we have to identify the relevant, irrelevant and marginal interactions; this is done by a power counting argument and it turns out, as it is standard in fermionic one dimensional systems, that the interactions quadratic in the fields are relevant, the quartic are marginal and the other processes are irrelevant. However there are in this model L many different terms bilinear or quartic in the fields, depending on the value of m in (16), and so it seems that there are L different non irrelevant interactions to be taken into account, which seems an hopeless task as we are interested in the $L \to \infty$ limit; this problem will be solved by a *improved power counting* in which the Diophantine condition plays a crucial role. Note finally that the quadratic interaction have a non trivial flow, so it is necessary to change the fermionic integration at each step; in other words the model has an *anomalous behaviour*, due to the fact that the model is close to a Luttinger liquid.

The integration is performed iteratively, setting $Z_0 = 1$, $\sigma_0 = \sigma$, in the following way: once that the fields $\psi^{(0)}, ..., \psi^{(h+1)}$ have been integrated we have

$$\int P_{Z_h}(d\psi^{(\le h)})\, e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})}. \tag{17}$$

Then, putting $C_h(\vec{k}')^{-1} = \sum_{j=-\infty}^{h} f_j(\vec{k}')$ and $\alpha(k') = (\cos k' - 1)\cos p_F$, $v_0 = \sin p_F$:

$$P_{Z_h}(d\psi^{(\leq h)}) = \prod_{\vec{k}'} \prod_{\omega=\pm 1} d\psi^{(\leq h)+}_{\vec{k}'+\omega\vec{p}_F,\omega} d\psi^{(\leq h)-}_{\vec{k}'+\omega\vec{p}_F,\omega} \tag{18}$$

$$\exp\Big\{ - \sum_{\omega=\pm 1} \int d\vec{k}' C_h(\vec{k}') Z_h \Big[\Big(-ik_0 - \alpha(k') + \omega v_0 \sin k' \Big) \psi^{(\leq h)+}_{\vec{k}'+\omega\vec{p}_F,\omega} \psi^{(\leq h)-}_{\vec{k}'+\omega\vec{p}_F,\omega} - \sigma_h(\vec{k}')\, \psi^{(\leq h)+}_{\vec{k}'+\omega\vec{p}_F,\omega} \psi^{(\leq h)-}_{\vec{k}'-\omega\vec{p}_F,-\omega} \Big] \Big\}.$$

Note that after $|h|$ steps the integration is different to the initial one; there is a wave function renormalization Z_h and a mass term σ_h. Moreover the effective potential at scale h has the form

$$\mathcal{V}^{(h)}(\psi^{(\leq h)}) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int d\vec{k}'_1 ... d\vec{k}'_{2n} \prod_{i=1}^{n} \psi^{\sigma_i(\leq h)}_{\vec{k}'_i+\omega_i\vec{p}_F,\omega_i} \delta(\sum_{i=1}^{2n} \sigma_i(\vec{k}'_i + \omega_i\vec{p}_F) + 2m\vec{p}_F) W^h_{2n,m}(\vec{k}'_1 + \vec{\omega}_1\vec{p}_F, ..; \{\omega\}). \tag{19}$$

In order to integrate $\psi^{(h)}$ we write $\mathcal{V}^{(h)}$ as $\mathcal{L}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)}$, with $\mathcal{R} = 1 - \mathcal{L}$. The $\mathcal{L}$ operation is defined to extract the non irrelevant terms in $\mathcal{V}^{(h)}$; it is easy to check from a power counting argument that the terms in $\mathcal{V}^{(h)}$ involving six or more fields are irrelevant, thus $\mathcal{L} = 0$ on such terms. Moreover, we will define $\mathcal{L} = 0$ on the addenda in (19) *not* verifying the condition

$$\sum_{i=1}^{2n} \sigma_i \omega_i p_F + 2mp_F = 0 \quad \text{mod. } 2\pi, \tag{20}$$

which means that we are considering irrelevant the terms such that the sum of momenta measured from the Fermi surface is not vanishing (but it can be arbitrary small, due to the irrationality of $\frac{p_F}{\pi}$). At the moment this definition of $\mathcal{L}$ is completely arbitrary; it will be clear in the next section where we will show that the terms non verifying (20) are indeed irrelevant (here is where the Diophantine condition plays a role).

In conclusion the definition of $\mathcal{L}$ is the following:

1) If $2n > 4$ then

$$\mathcal{L}W^h_{2n,m}(\vec{k}_1, ...) = 0$$

2) If $2n = 4$ then

$$\mathcal{L}W^h_{4,m}(\vec{k}_1, ...) = \delta_{m,0}\delta_{\sum_{i=1}^4 \sigma_i\vec{\omega}_i,0} W^h_{4,m}(\omega_1\vec{p}_F, ..., \omega_4\vec{p}_F) \tag{21}$$

3)If $2n = 2$, $\omega_1 = \omega_2$ then

$$\mathcal{L}\{W^h_{2,m}(\vec{k}'_1 + \omega_1\vec{p}_F, \vec{k}'_2 + \omega_2\vec{p}_F) = \delta_{m,0}[W^h_{2,m}(\omega_1\vec{p}_F, \omega_2\vec{p}_F)$$
$$+\omega_1 E(k' + \omega_1 p_F)\partial_k W^h_{2,m}(\omega_1\vec{p}_F, \omega_2\vec{p}_F) + k^0\partial_{k_0} W^h_{2,m}(\omega_1\vec{p}_F, \omega_2\vec{p}_F)], \tag{22}$$

where $E(k' + \omega p_F) = v_0\omega \sin k' + (1 - \cos k')\cos p_F$ and the symbol $\partial_k, \partial_{k_0}$ means discrete derivatives.

4)If $2n = 2$, $\omega_1 = -\omega_2$ then

$$\mathcal{L}W^h_{2,m}(\vec{k}'_1 + \omega_1\vec{p}_F, \vec{k}'_2 + \omega_2\vec{p}_F) = \delta_{m,\omega_2} W^h_{2,m}(\omega_1\vec{p}_F, \omega_2\vec{p}_F). \tag{23}$$

The Kronecker deltas in the r.h.s. of (21), (22)(23) ensure that $\mathcal{L} = 0$ if (20) is not verified.

We find

$$\mathcal{L}\mathcal{V}^{(h)}(\psi) = \gamma^h n_h + F_\nu^{(\leq h)} + s_h F_\sigma^{(\leq h)} + z_h F_\zeta^{(\leq h)} + a_h F_\alpha^{(\leq h)} + u_h F_U^{(\leq h)}, \tag{24}$$

where

$$F_\sigma^{(\leq h)} = \sum_{\omega=\pm 1} \int d\vec{k}' \psi^{(\leq h)+}_{\vec{k}'+\omega\vec{p}_F,\omega} \psi^{(\leq h)-}_{\vec{k}'-\omega\vec{p}_F,-\omega},$$
$$F_i^{(\leq h)} = \sum_{\omega=\pm 1} \int d\vec{k}' f_i(\vec{k}') \psi^{(\leq h)+}_{\vec{k}'+\omega\vec{p}_F,\omega} \psi^{(\leq h)-}_{\vec{k}'+\omega\vec{p}_F,\omega}, \tag{25}$$
$$F_U^{(\leq h)} = \int [\prod_{i=1}^4 d\vec{k}'_i] \delta(\sum_{i=1}^4 \sigma_i\vec{k}_i) \psi^{(\leq h)+}_{\vec{k}'_1+\vec{p}_F,1} \psi^{(\leq h)-}_{\vec{k}'_2+\vec{p}_F,1} \psi^{(\leq 0)+}_{\vec{k}'_3-\vec{p}_F,-1} \psi^{(\leq 0)-}_{\vec{k}'_4-\vec{p}_F,-1} \delta(\sum_{i=1}^4 \sigma_i\vec{k}_i),$$

where $i = \nu, \zeta, \alpha$ and $f_\nu = 1$, $f_\zeta = -ik_0$ and $f_\alpha = E(k' + \omega p_F)$; moreover $u_0 = U(\hat v(0) - \hat v(2p_F)) + O(U^2)$, $s_0 = O(U\lambda)$, $a_0, z_0 = O(U^2)$, $n_0 = \nu + O(U)$. Note that in $\mathcal{L}V^{(h)}$ there are terms renormalizing mass and the wave function renormalization and it is convenient to include them in the fermionic integration writing

$$\int P_{Z_h}(d\psi^{(\le h)})\, e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})} = \int \tilde P_{Z_{h-1}}(d\psi^{(\le h)})\, e^{-\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})} \tag{26}$$

where $\tilde P_{Z_{h-1}}(d\psi^{(\le h)})$ is defined as $P_{Z_h}(d\psi^{(\le h)})$ eq(18) with Z_{h-1} and σ_{h-1} replacing Z_h, σ_h, with

$$Z_{h-1}(\vec k') = Z_h(1 + C_h^{-1}(\vec k') z_h); \qquad Z_{h-1}(\vec k')\sigma_{h-1}(\vec k') = Z_h(\sigma_h(\vec k') + C_h^{-1}(\vec k') s_h. \tag{27}$$

Moreover $\tilde{\mathcal{V}}^{(h)} = \mathcal{L}\tilde{\mathcal{V}}^{(h)} + (1-\mathcal{L})\mathcal{V}^{(h)}$ and

$$\mathcal{L}\tilde{\mathcal{V}}^{(h)} = \gamma^h n_h F_\nu^{(\le h)} + (a_h - z_h) F_\alpha^{(\le h)} + u_h F_U^{(\le h)} \tag{28}$$

The r.h.s of (26) can be written as

$$\int P_{Z_{h-1}}(d\psi^{(\le h-1)}) \int \tilde P_{Z_{h-1}}(d\psi^{(h)})\, e^{-\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})} \tag{29}$$

where $P_{Z_{h-1}}(d\psi^{(\le h-1)})$ and $\tilde P_{Z_{h-1}}(d\psi^{(h)})$ are given by (18) with Z_{h-1} replaced by $Z_{h-1}(0) \equiv Z_{h-1}$ and $C_h(\vec k')$ replaced with $C_{h-1}(\vec k')$ and $\tilde f_h^{-1}(\vec k')$ respectively, if

$$\tilde f_h(\vec k') = Z_{h-1}[\frac{C_h^{-1}(\vec k')}{Z_{h-1}(\vec k')} - \frac{C_{h-1}^{-1}(\vec k')}{Z_{h-1}}] \tag{30}$$

and $\psi^{(\le h)}$ replaced with $\psi^{(\le h-1)}$ and $\psi^{(h)}$ respectively. Note that $\tilde f_h(\vec k')$ is a compact support function, with support of width $O(\gamma^h)$ and far $O(\gamma^h)$ from the "singularity" *i.e.* ωp_F. The Grassmanian integration $\tilde P_{Z_{h-1}}(d\psi^{(h)})$ has propagator

$$g^h_{\omega,\omega'}(\vec x - \vec y) = \int \tilde P_{Z_{h-1}}(d\psi^{(h)}) \psi^-_{\omega,\vec x}\psi^+_{\omega',\vec y}$$

given by

$$\frac{1}{Z_{h-1}} \int d\vec k' e^{-i\vec k'(\vec x - \vec y)} \frac{1}{A_{h-1}(k')} \tilde f_h(\vec k') \begin{pmatrix} -ik_0 - \alpha(k') - v_0 \sin k' & \sigma_{h-1}(k') \\ \sigma_{h-1}(k') & -ik_0 - \alpha(k') + v_0 \sin k' \end{pmatrix}, \tag{31}$$

where $A_h(\vec k') = [-ik_0 - \alpha(k')]^2 - (v_0 \sin k')^2 - [\sigma_{h-1}(\vec k')]^2$. It is convenient to write decompose the propagator as

$$g^{(h)}_{\omega,\omega}(\vec x - \vec y) = g^{(h)}_{L,\omega}(\vec x - \vec y) + C_2^{(h)}(\vec x - \vec y), \tag{32}$$

where

$$g^{(h)}_{L,\omega}(\vec x - \vec y) = \frac{1}{L\beta} \sum_{\vec k'} \frac{e^{-i\vec k'(\vec x - \vec y)}}{-ik_0 + \omega v_0 \sin k' + \alpha(k')} \tilde f_h(\vec k') \tag{33}$$

and, for any integer $N > 1$

$$|g^{(h)}_{L,\omega}(\vec x - \vec y)| \le \frac{\gamma^h C_N}{1 + (\gamma^h |\vec x - \vec y|)^N}, \qquad |C_2^{(h)}(\vec x - \vec y)| \le |\frac{\sigma_h}{\gamma^h}|^2 \frac{\gamma^h C_N}{1 + (\gamma^h |\vec x - \vec y|)^N}. \tag{34}$$

Moreover

$$|g^{(h)}_{\omega,-\omega}(\vec x - \vec y)| \le |\frac{\sigma_h}{\gamma^h}| \frac{\gamma^h C_N}{1 + (\gamma^h |\vec x - \vec y)|)^N} \tag{35}$$

Finally we *rescale* the fields so that

$$\int P_{Z_{h-1}}(d\psi^{(\le h-1)}) \int \tilde{P}_{Z_{h-1}}(d\psi^{(h)})\, e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})} \tag{36}$$

where

$$\mathcal{L}\hat{\mathcal{V}}^{(h)}(\psi) = \gamma^h \nu_h F_\nu^{(\le h)} + \delta_h F_\alpha^{(\le h)} + U_h F_U^{(\le h)}, \tag{37}$$

and by definition

$$\nu_h = \frac{Z_h}{Z_{h-1}} n_h; \quad \delta_h = \frac{Z_h}{Z_{h-1}}(a_h - z_h); \quad U_h = (\frac{Z_h}{Z_{h-1}})^2 u_h. \tag{38}$$

We perform the integration

$$\int \tilde{P}_{Z_{h-1}}(d\psi^{(h)})\, e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})} = e^{-\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)})}, \tag{39}$$

where $\mathcal{V}^{(h-1)}$ has the same form as $\mathcal{V}^{(h)}$ and the procedure can be iterated, as the insertion of (39) in (29) gives an expression like (17) with $h-1$ replacing h. The above procedure is iterated untill a scale h^* defined as the minimum h such that $\gamma^h > |\sigma_h|$ is reached. Then we will integrate directly the field $\psi^{(<h^*)} = \sum_{k=-\infty}^{h^*} \psi^{(k)}$ without splitting the corresponding integration in scales (as was done for $h > h^*$). This can be done as $g^{\le h^*}(\vec{x} - \vec{y})$ verifies the bound eq(34) with h^* replacing h *i.e.* it verifies the bound valid for a single scale; the reason is thatfor momenta larger than $O(\gamma^{h^*})$ the theory it is essentially a massless theory and for momenta smaller is a massive theory with mass $O(\gamma^{h^*})$. We will call *running coupling constants* $\vec{v}_h = (U_h, \delta_h, \nu_h)$ and *renormalization constants* Z_h, σ_h; their behaviour as a function of h can be found by an iterative equation called *beta function*. Note that the irrelevant terms are *not* neglected, contrary to what is done in the usual RG methods, which are only approximative and not mathematically exact. The expansion generated by our RG is instead *exact* in a mathematical sense and nothing is neglected (see Refs. [24] and [16] for details).

B. The small divisor problem

We have considered *irrelevant* the terms involving two or four fermions in the effective potential not verifying (20). Looking at (19) we see that each addend contributing to the effective potential describes the interactions of $2n$ fermions whose momenta *measured from the Fermi surface* verify

$$\sum_{i=1}^{2n} \sigma_i k_i' = \sum_i \sigma_i \omega_i p_F + 2mp_F. \tag{40}$$

Then (20) says simply that the non irrelevant terms are only the ones in which the sum of the momenta measured from the Fermi surface is vanishing modulo 2π. This condition seems very natural in the *commensurate* case *i.e.* when $p_F = \pi \frac{P}{Q}$; in that case, for $n = 1, 2$ if the r.h.s. of (20) is non vanishing modulo 2π, then it is greater than $O(\frac{1}{Q})$ so for Q not too big at least one fermion involved has a momentum far enough from the Fermi surface. However things are not so simple in *incommensurate* case; in such case for $n = 1, 2$ the r.h.s. of (19) can be very small (modulo 2π) for very large m; in other words there are terms in the effective potential which are dimensionally relevant or marginal involving fermions with momenta arbitrarily close to the Fermi surface and not verifying (20); for instance $\psi^+_{k+p_F,1}\psi^-_{k-p_F+2mp_F,-1}$ with $2mp_F + 2k\pi \simeq 0$ for a suitable k. Then in the incommensurate case it is not clear if the terms not verifying (20) are really irrelevant (this problem is often not seen in literature, see for intance Ref. [31]) . This problem, with an interely different language, is well known in classical or celestial mechanics as the *small divisor problem*, for instance in the KAM or Lindstedt series for invariant tori of an hamiltonian system close to an integrable one. It is possible to write such classical series in terms of Feynman graphs so that the similarity becomes very clear, see Refs. [12] and [14]; the crucial difference is that such graphs have no loops, contrary to what happens here. Another remarkable case in which small divisors appear is in the study of Schroedinger equation with a quasi periodic potential (very related to our problem in the $U = 0$ case); if a *Diophantine condition* is assumed on the period there are quasi-Bloch states if λ is small, see Ref. [11], for suitable values of the quasi momentum, while the eigenstates are localized (*Anderson localization*) for large λ, see Ref. [29].

The fact that the contributions to the effective potential not verifying (20) are *irrelevant* in a RG sense means that the perturbative series as function of $\lambda_k, \delta_k, \nu_k$ and $\frac{Z_k}{Z_{k-1}}$ and $\frac{\sigma_k}{\sigma_{k-1}}$ are convergent in a neighborhood of the origin.

To give a complete mathematical proof of the above statement is not straightforward, as one has to use determinant bounds for the fermionic truncated expectation; one cannot simply prove that each Feynman graph admits a finite bound as the number of Feynman graphs at order n is $O(n!^2)$, so we refer to Ref. [24]. However the key idea why the terms not verifying (20) are irrelevant can be understood from an analysis based on Feynman graphs.

Each $W^{(h)}_{2n,m}$ admits an expansion in terms of *Feynman diagrams* defined in the following way. A k-th order diagram contributing to $W^{(h)}_{2n,m}$ can be obtained from k graph-elements representing the addenda in (37) or in $\mathcal{RV}^{(0)}$ (16) by pairing the half lines (bearing indices $h, \omega, \vec{k}'$). The unpaired lines are called *external lines*, and to each paired line we associate a propagator $g^{h_i}_{\omega_i,\omega'_i}(\vec{k}'_i)$ (31); integrating the product of these factors over all the momenta $\vec{k}'_i$ of the paired lines we obtain the *value* of the graph, if the expression is multiplied by a suitable sign to take into account the Fermi statistic. A maximal connected subset of lines with scales $\geq h_v$ is called *cluster* with scale h_v, and denoted by v. An inclusion relation can be established between the clusters, in such a way that the innermost clusters are the clusters with the higher scale, and so on; see the picture for an example of graphs with its clusters, pictured as boxes including the paired lines. The half-lines (contracted or not contracted) are emerging by the *end-points*, associated to $\vec{v}_h$ or to the kernels of $\mathcal{RV}^0$; if to an end-point is associated $\vec{v}_k$, the minimal cluster containing it has scale k.

To each Feynman graph it is associated by the above rule a *value*, and so to each subdiagram associated to each cluster; moreover, to the lines external to a cluster is associated a momentum $\vec{k}'$ smaller in modulus than one flowing in the lines internal to a cluster. We denote by v each cluster, h_v its scale (*i.e.* all the lines internal to the cluster v have scale $\leq h_v$ and at least one has scale h_v, and the external are larger), by P_v the indices of the external lines, by $|P_v|$ their number and by $N_v p_F$ the sum of the momenta $\vec{k}$ of the external lines. If v' is the minimal cluster enclosing the cluster v, the $\mathcal{R} \neq 1$ operation produces one or more $\vec{k}'$ associated with the external lines (which are bounded by $\gamma^{h_{v'}}$) and one or more derivatives on the propagators internal to the cluster (giving one or more extra γ^{-h_v}). At the end (see for instance Ref. [16]) one gets the following bound (essentially found by power counting using (34) for bounding the propagators) for a graph with k vertices and $2n$ external lines with value $G^{(h)}_{2n,k}$

$$|G^{(h)}_{2n,k}| \leq C^k \varepsilon^k \gamma^{(2-\frac{|2n|}{2})h} \prod_i |\hat{\varphi}_{n_i}| \prod_v \gamma^{-(-2+\frac{|P_v|}{2}+z_v)(h_v - h_{v'})}, \tag{41}$$

where $z_v = 1$ if $|P_v| = 4$ and $N_v = 0$; $z_v = 1$ if $|P_v| = 2$ and $N_v = \omega_2$, $\omega_1 = -\omega_2$; $z_v = 2$ if $|P_v| = 2$, $N_v = 0$; $z_v = 0$ in all the remaining cases. The factor z_v is due to the action on $\mathcal{R}$ on each cluster v. In order to sum over all the possible assignments of labels h_v we need that $-2 + \frac{|P_v|}{2} + z_v > 0$ and this is true for all the clusters with two or four external lines not verifying (20) (except when $|P_v| = 2$ and $\omega_1 = -\omega_2$ in which case it is zero). We have to improve the bound in the cases in which $-2 + \frac{|P_v|}{2} + z_v \leq 0$. The first case we can consider is $|P_v| = 2$ and $\omega_1 = -\omega_2$, $N_v = \omega_2$. A similar cluster can be produced only if a)there is a non diagonal propagator $g^{(k)}_{\omega,-\omega}$ internal to the cluster v; b)if there is one or more points associated to $\hat{\varphi}_n$ internal to v. In both cases this means that there is in the bounds an extra factor $\sigma_k \gamma^{-k}$ for some scale $k > h_v$, see (35), and

$$|\sigma_k|\gamma^{-k} = \frac{|\sigma_k|}{|\sigma_h|}\gamma^{h-k}\frac{|\sigma_h|}{\gamma^h} \leq \frac{|\sigma_h|}{\gamma^h}\gamma^{\frac{1}{2}(h-k)} \tag{42}$$

and the factor $\gamma^{\frac{1}{2}(h-k)}$ allows us to sum over h_v.

It remains to discuss the clusters with $|P_v| = 2, 4$ not verifying (20); here is where the Diophantine condition comes in. Given a cluster v, if $\sum_i \sigma_i \omega_i p_F + 2N_v p_F \neq 0$ mod. 2π then

$$|N_v| \geq C[\frac{\gamma^{-\frac{h_{v'}}{\tau}}}{|P_v|^{\frac{1}{\tau}}} - |P_v|]. \tag{43}$$

In fact, by the compact support properties of the propagators and the *Diophantine condition*

$$a_0 \gamma^{h_{v'}} \geq ||\sum_{i=1}^{|P_v|} \sigma_i k'_i||_T \geq ||2N_v p_F + \sum_{i=1}^{|P_v|} \sigma_i \omega_i p_F||_T \geq C_0 (2|P_v| + |N_v|)^{-\tau}. \tag{44}$$

The meaning of (43) is quite clear; remembering that the momenta of the external lines is $O(\gamma^{h_{v'}})$, (43) says that the external fields can have momenta very close to the Fermi surface only if N_v is very large. The corresponding contribution is then very small, for the exponential decay properties of $\hat{\varphi}_n$. We can define a depth D_v defined in the following way: if v does not contain any cluster $D_v = 1$ otherwise $D_v = 1 + max_{v''} D_{v''}$ where v'' are the clusters contained in v. It is easy to see that

$$\prod_i |\hat{\varphi}_{n_i}| \leq e^{\frac{-k|m|}{4}} \prod_i e^{\frac{-k|n_i|}{4}} \prod_v e^{-\frac{k|N_v|}{2^{D_v+1}}}. \tag{45}$$

Using (43) and the fact that $D_v \leq -h_{v'} + 2$ we get

$$\prod_i |\hat{\varphi}_{n_i}| \leq e^{\frac{-k|m|}{4}} \prod_i e^{\frac{-k|n_i|}{4}} \prod_v{}^{*} e^{-\frac{k\gamma^{\frac{-h_{v'}}{\tau}}}{2^{-h_{v'}+3}}}, \tag{46}$$

where $\prod_v^*$ is restricted to clusters with two or four external lines with $N_v \neq 0$; choosing $\gamma^{\frac{1}{\tau}} 2^{-1} > 1$ we can associated to each of this clusters a factor

$$\exp[-k\gamma^{\frac{-h_{v'}}{\tau}} 2^{h_{v'}+3}] < \gamma^{3h_{v'}} \leq \gamma^{3(h_{v'}-h_v)} \tag{47}$$

and this extra factor allows us to sum over the scale indices.

In the above discussion of small divisor problem we have used the exponential decay of $\hat{\varphi}_n$ but such condition could be probably relaxed to a power law decay *i.e.* $|\hat{\varphi}_n| \leq \frac{C_N}{|n|^N}$ for some integer N *i.e.* relaxing the analyticity condition for φ_x to a differentiability one. This is quite reasonable, by analogy with KAM theorem which is valid not only for analytic but also for differentiable perturbations, see Ref. [26], but the analysis would be probably much more involved.

C. The flow of the running coupling constants

The equations for the running coupling constants are, for $h \geq h^*$

$$\begin{aligned} &\nu_{h-1} = \gamma \nu_h + G^h_\nu \quad U_{h-1} = U_h + G^h_U \\ &\sigma_{h-1} = \sigma_h + G^h_\sigma \quad \delta_{h-1} = \delta_h + G^h_\delta \\ &\frac{Z_{h-1}}{Z_h} = 1 + G^h_z \end{aligned} \tag{48}$$

It is convenient to split $G^{(h)}_i$, with $i = \mu, \sigma, \nu$ into

$$G^h_i(\mu_h, \nu_h, \sigma_h; ...; \mu_0, \nu_0, \sigma_0) = G^{1,h}_i(\mu_h, \nu_h; ...; \mu_0, \nu_0) + G^{2,h}_i(\mu_h, \nu_h, \sigma_h; ...; \mu_0, \nu_0, \sigma_0) \tag{49}$$

where we have spitted $g^{(h)}_{\omega,\omega}$ as in (32) and $G^{1,h}_i$ contains no non-diagonal propagators and only the part $g^{(h)}_{L;\omega,\omega}$ of the diagonal propagators; moreover, there are no vertices with $m \neq 0$; in $G^{2,h}_i$ are all the remaining contributions. It is easy to check that for $i = \mu, \nu$, for $max_{k \geq h} |\vec{v}_k| \leq \varepsilon$,

$$|G^{2,h}_i| \leq C[\frac{\sigma_h}{\gamma^h}]^2 \varepsilon^2. \tag{50}$$

This follows from the bound (34) for C_2^h and from the fact that ν, μ are momentum conserving terms. For $i = \sigma$, by symmetry reasons, $G_i^{1,h} \equiv 0$ and

$$|G_\sigma^{2,h}(\mu_h, \nu_h, \sigma_h; ...; \mu_0, \nu_0, \sigma_0)| \le C|U_h \sigma_h|. \tag{51}$$

We decompose, for $i = \mu, \nu$,

$$G_i^{1,h}(\mu_h, \nu_h; ...; \mu_0, \nu_0) = \bar{G}_i^{1,h}(\mu_h; ...; \mu_0) + \hat{G}_i^{1,h}(\mu_h, \nu_h; ...; \mu_0, \nu_0) \tag{52}$$

where the first term in the r.h.s. of (52) is obtained putting $\nu_k = 0$, $k \ge h$ in the l.h.s. It is easy to see, from the fact that $g_{L,\omega}^{(h)}(\vec{x}; \vec{y})$ can be divided in a even part plus a correction smaller than a factor $\gamma^{\frac{h}{4}}$, that for $max_{k \ge h} |\vec{v}_k| \le \varepsilon$

$$|\bar{G}_\nu^{1,h}(\mu_h; ...; \mu_0) \le C\varepsilon \gamma^{\frac{h}{4}}. \tag{53}$$

On the other hand

$$|\bar{G}_\mu^{1,h}(\mu_h; ...; \mu_h)| \le C\varepsilon^2 \gamma^{\frac{h}{2}}, \tag{54}$$

as one can prove using the exact solution of the Luttinger model, see Refs. [9], [7],and [17]. Moreover we have that, for $i = \nu, \mu$

$$|\hat{G}_i^{1,h}(\mu_h, \nu_h; ...; \mu_0, \nu_0)| \le C\nu_h |U_h|^2 \tag{55}$$

Finally by a second order computation one obtains

$$G_\sigma^{1,h} = \sigma_h U_h [\beta_1 + \bar{G}_\sigma^{1,h}], \qquad G_z^{1,h} = U_h^2 [\beta_2 + \bar{G}_z^{1,h}], \tag{56}$$

with β_1, β_2 non vanishing positive constants and $|\bar{G}_\sigma^{1,h}| \le C|U_h|$, and $|\bar{G}_z^{1,h}| \le C|U_h|$.

By using the above properties we can control the flow of the running coupling constants. In fact, if $|\nu_k| \le C\varepsilon[\gamma^{\frac{k}{4}} + \frac{|\sigma_k|}{\gamma^k}]$ for any $k \ge h^*$ (which will be proved in the following section), it follows that there exist positive constants c_1, c_2, c_3, c_4, C such that, if λ, u are small enough and $h \ge h^*$:

$$|U_{h-1} - U| < CU^{3/2}; \qquad e^{-U\beta_1 c_3 h} < \frac{|\sigma_{h-1}|}{|\sigma_0|} < e^{-U\beta_1 c_4 h}; \qquad e^{-\beta_2 c_1 U^2 h} < Z_{h-1} < e^{-\beta_2 c_2 U^2 h}. \tag{57}$$

D. Determination of the counterterm ν

We show that it is possible to fix ν to a λ-independent value; more exactly we show that it is possible to choose ν as in the $\lambda = 0$ case so that ν_k is small for any $k \ge h^*$. In the $\lambda = 0$ case, $\sigma_k = 0$ and there are no contribution to the effective potential with $m \ne 0$; calling $\tilde{\nu}_k, \tilde{\mu}_k$ the analogous of ν_k, μ_k, we can write

$$\tilde{\nu}_h = \gamma^{-h+1} [\nu + \sum_{k=h+1}^{1} \gamma^{k-2} G_\nu^{1,k}(\tilde{\nu}, \tilde{\mu})], \tag{58}$$

where $G_\nu^{1,k}(\tilde{\nu}, \tilde{\mu}) = G_\nu^{1,k}(\tilde{\nu}_k, \tilde{\mu}_k; ....; \tilde{\nu}_0, \tilde{\mu}_0)$. We choose

$$\nu = -\sum_{k=-\infty}^{1} \gamma^{k-2} G_\nu^{1,k}(\tilde{\nu}, \tilde{\mu}) \tag{59}$$

then

$$\tilde{\nu}_h = -\gamma^{-h} \sum_{k=-\infty}^{h} \gamma^{k-1} G_\nu^{1,k}(\tilde{\nu}, \tilde{\mu}) \tag{60}$$

and $|\tilde{\nu}_h| \le C\varepsilon \gamma^{\frac{h}{4}}$, as by (53)

$$\gamma^{-h}\sum_{k=-\infty}^{h}\gamma^{k-1}|G^{1,k}_{\nu}(\tilde\nu,\tilde\mu)|\le C'\varepsilon\gamma^{-h}\sum_{k=-\infty}^{h}\gamma^{k}\gamma^{\frac14 k}\le C\varepsilon\gamma^{\frac{h}{4}}. \tag{61}$$

For the model with $\lambda\neq 0$, for $h\ge h^*$

$$\nu_h=\gamma^{-h+1}[\nu+\sum_{k=h+1}^{1}\gamma^{k-2}[G^{2,k}_{\nu}(\nu,\mu,\sigma)+G^{1,k}_{\nu}(\nu,\mu)]] \tag{62}$$

and, inserting ν given by (59),

$$\nu_h-\tilde\nu_h=\gamma^{-h+1}\{\sum_{k=h+1}^{1}\gamma^{k-2}G^{2,k}_{\nu}(\nu,\mu,\sigma)+\sum_{k=h+1}^{1}\gamma^{k-2}[G^{1,k}_{\nu}(\nu,\mu)-G^{1,k}_{\nu}(\tilde\nu,\tilde\mu)]\} \tag{63}$$

We prove that, for $h\ge h^*$,

$$|\nu_h-\tilde\nu_h|\le\varepsilon\bar C(\frac{\sigma_h}{\gamma^h})^2,\qquad |\tilde\mu_h-\mu_h|\le\varepsilon(\frac{\sigma_h}{\gamma^h})^2. \tag{64}$$

The proof is done by induction, assuming that it holds for scales $\ge h+1$ and proving by eq(63) that it holds for scale h. Looking at the first sum in (63) and using (50),(57)

$$\gamma^{-h}\sum_{k=h+1}^{1}\gamma^{k-2}|G^{2,k}_{\nu}(\nu,\mu)|\le C_1\gamma^{-h}\varepsilon^2\sum_{k=h+1}^{1}\gamma^{k-2}(\frac{\sigma_k}{\gamma^k})^2$$

$$\le(\frac{\sigma_h}{\gamma^h})^2C_2\varepsilon^2\sum_{k=h+1}^{1}\gamma^{h-k}(\frac{\sigma_k}{\sigma_h})^2\le C_3\varepsilon^2(\frac{\sigma_h}{\gamma^h})^2. \tag{65}$$

Finally we can write $G^{1,k}_{\nu}(\nu,\mu)-G^{1,k}_{\nu}(\tilde\nu,\tilde\mu)=\sum_{\bar k>k}D_{\bar k,k}$, with

$$D_{\bar k,k}=G^{1,k}_{\nu}(\nu_k,\mu_k;...;\nu_{\bar k},\mu_{\bar k};\tilde\nu_{\bar k+1},\tilde\mu_{\bar k+1};...;\tilde\nu_0,\tilde\mu_0)-G^{1,k}_{\nu}(\nu_k,\mu_k;...;\tilde\nu_{\bar k},\tilde\mu_{\bar k};\tilde\nu_{\bar k+1},\tilde\mu_{\bar k+1};...;\tilde\nu_0,\tilde\mu_0) \tag{66}$$

and, by the inductive hypothesis and (41)

$$\sum_{\bar k\ge k}|D_{\bar k,k}|\le C_1\bar C\varepsilon^2\sum_{\bar k\ge k}\gamma^{\frac12(k-\bar k)}(\frac{\sigma_{\bar k}}{\gamma^{\bar k}})^2\le C_2\bar C\varepsilon^2(\frac{\sigma_k}{\gamma^k})^2, \tag{67}$$

so that the last sum in eq(63) is bounded by $\bar C\varepsilon(\frac{\sigma_h}{\gamma^h})^2$ with $\bar C=4C$, and (64) is proved; from that equation it follows that for $h\ge h^*$, then $|\nu_h|\le C\varepsilon$, so that it is possible to have the flow of ν_h is bounded choosing a λ independent ν.

E. Bounds for the density

An expansion similar to the one we have seen for the effective potential can be defined for the Schwinger function and hence for $\hat\rho_n$. One obtains

$$\hat\rho_1=\sum_{h=h^*}^{1}\int d\vec k\hat g^{(h)}_{1,-1}(\vec k)+\sum_{k=1}^{\infty}\sum_{h=h^*}^{1}\hat\rho^{k,(h)}_{1,-1} \tag{68}$$

with $\hat\rho^{k,(h)}_{1,-1}$ being sum over Feynman graphs similar to the one for the effective potential in which h is the lowest scale of the propagators and k the number of points. There are two kinds of contributions. The ones in which there are no points associated to $\lambda\hat\varphi_n$, which means that there is at least a non diagonal propagator; such terms are bounded by $C^k\varepsilon^k|\sigma_h|Z_h^{-1}$. The remaining ones (in which there is at least an irrelevant point) are bounded by $C^k\varepsilon^k\gamma^{\frac{h}{4}}$, the factor $\gamma^{\frac{h}{4}}$ coming from (41).

In the same way for $n\neq 1$ $\hat\rho_n=\sum_{k=1}^{\infty}\sum_{h=h^*}^{1}\hat\rho^{k,(h)}_n$. To the expansion of $\hat\rho_n$ with $|n|>1$ are contributing only graphs with at least one irrelevant point $\lambda\hat\varphi_{n'}$; in fact it is not possible, by conservation of momenta, to get a contribution to $\hat\rho_n$ with $|n|>1$ from graphs containing only U,δ,ν vertices (taking into account that there are non diagonal propagators, the difference of the external momenta of such graphs can be at most $0,\pm 2p_F$). Hence $|\rho^{k(h)}_n|\le e^{-\kappa/2|n|}|\sigma|C^k\varepsilon^k\gamma^{\frac{h}{4}}$.

This concludes the construction of a well defined algorithm for computing the l.h.s. of (5) with any prefixed precision, if λ, U are small enough; in the next section we try to find a solution $\varphi\in\Omega$ of (5) or (6) by a contraction method.

III. SOLUTION OF THE SELF CONSISTENCE EQUATION

A. Contraction mapping

We have seen in §2.E that $\hat\rho_n = \delta_{n,\pm1}\hat\rho_n^{(0)} + \sum_{k=0}^{\infty}\hat\rho_n^{(k)}$, where $\hat\rho_n^{(0)}$ is the first addend in (68) and $\hat\rho_n^{(1)}$ is given by graphs with one point associated to an irrelevat term $\lambda\hat\varphi_{n'}$. We keep in the r.h.s. of (5), instead of the full expansion, just the terms with $k = 0, 1$, forgetting the others for the moment.

We find natural, in order to find a solution of (6), to proceed in the following way, calling $\Phi_n = \lambda\hat\varphi_n$ for $|n| > 1$:

1)We consider $\sigma \equiv \lambda\hat\varphi_1$ as a parameter and we study $\Phi_n = \lambda^2\hat\rho_n^{(1)}(\lambda, U, \sigma, \Phi)$, looking for a solution $\Phi(\lambda, U, \sigma)$ in the class Ω.

2)If a non trivial solution $\Phi(\lambda, U, \sigma)$ is found, we insert it in (6) with $n = 1$ looking for a solution σ function of λ, U.

According to the above strategy, we have to solve, for $|n| > 1$

$$\hat\varphi_n = -\lambda^2 c_n^{(1)}(\sigma)\hat\varphi_n + \lambda\hat\rho_n^{(1)}(\lambda, U, \sigma, \Phi), \tag{69}$$

where

$$\begin{aligned} c_n^{(1)}(\sigma) = \int d\vec k \Big\{ &\tilde g_{1,1}^{(1)}(\vec k')\, \tilde g_{1,1}^{(1)}(\vec k' + 2n\vec p_F) + \sum_{\omega=\pm1}\Big[\tilde g_{1,1}^{(1)}(\vec k') \sum_{h=h^*}^{0} \frac{1}{Z_h}\tilde g_{\omega,\omega}^{(h)}(\vec k' + 2n\vec p_F + (1-\omega)\vec p_F) \\ &+ \sum_{h=h^*}^{0}\frac{1}{Z_h}\tilde g_{\omega,\omega}^{(h)}(\vec k')\, \tilde g_{1,1}^{(1)}(\vec k' + 2n\vec p_F - (1-\omega)\vec p_F) + \sum_{h=h^*}^{0}\frac{1}{Z_h}\tilde g_{\omega,-\omega}^{(h)}(\vec k') \sum_{h'=h^*}^{0}\frac{1}{Z_{h'}}\tilde g_{-\omega,\omega}^{(h')}(\vec k' + 2n\vec p_F) \\ &+ \sum_{h=h^*}^{0}\frac{1}{Z_h}\tilde g_{\omega,\omega}^{(h')}(\vec k') \sum_{h'=h^*}^{0}\frac{1}{Z_{h'}}\tilde g_{\omega,\omega}^{(h')}(\vec k' + 2n\vec p_F) + \sum_{h=h^*}^{0}\frac{1}{Z_h}\tilde g_{\omega,\omega}^{(h)}(\vec k') \sum_{h'=h^*}^{0}\frac{1}{Z_{h'}}\tilde g_{-\omega,-\omega}^{(h')}(\vec k' + (2n+2\omega)\vec p_F)\Big]\Big\} \end{aligned} \tag{70}$$

and

$$\hat\rho_n^{(1)} = \sum_{h,h'=h^*}^{1}\frac{1}{Z_h Z_{h'}}\sum_{m\neq0,n}\sum_{\substack{\omega_1,\omega_1'\\ \omega_2,\omega_2'}}\delta_{2m-\omega_1+\omega_1'-\omega_2+\omega_2',2n}\lambda\hat\varphi_m\int d\vec k' g_{\omega_1,\omega_1'}^{(h)}(\vec k') g_{\omega_2,\omega_2'}^{(h')}(\vec k' + (2m+\omega_1'-\omega_2)p_F), \tag{71}$$

where $g^{(1)}(\vec k) = g^{u.v.}(\vec k) = g_{\omega,\omega}^{(1)}(\vec k')$ if $\vec k = \vec k' + \omega p_F$. We shall rewrite (69) as

$$\Phi_n = \frac{\lambda^2}{1 + \lambda^2 c_n^{(1)}}\hat\rho_n^{(1)}(\lambda, U, \sigma, \Phi). \tag{72}$$

and we will look for a solution $\Phi(\lambda, U, \sigma)$ of such equation by applying an iterative method.

For a fixed L, Φ is a finite sequence of $L-3$ elements, which can be thought as a vector in R^{L-3}. We consider the space $\mathcal{F} = C^1(R^{L-3})$ of C^1-functions of $\hat\varphi_n$, $n = 2, 3, ..., L_i - 1$. We shall define a norm in $\mathcal{F}$ for any λ, σ different from zero

$$\|\Phi\|_{\mathcal{F}} = \sup_{|n|>1}\left\{ e^{\frac{|n||\log|\lambda||}{10}}[|\sigma|^{-1}|\Phi_n(\sigma)| + |\frac{\partial\Phi_n}{\partial\sigma}|]\right\}. \tag{73}$$

We shall also define

$$\mathcal{B} = \{\Phi \in \mathcal{F} : \|\Phi\|_{\mathcal{F}} \le 1\};$$

The solutions of (72) can be seen as fixed points of the operator $\mathbf{T}_\lambda : \mathcal{F} \to \mathcal{F}$, defined by the equation:

$$[\mathbf{T}_\lambda(\Phi)]_n(\sigma) = \frac{\lambda^2\tilde\rho_n^{(1)}(\lambda, U, \sigma, \Phi)}{(1 + \lambda^2 c_n^{(1)}(\sigma))}, \tag{74}$$

In the sum over m in (71) $m \neq n$ so that there is at least a non-diagonal propagator, so that by (57), if $\omega_1' = -\omega_1$ (say),

$$\sum_{h,h'=h^*}^{1} \frac{1}{Z_h Z_{h'}} | \int d\vec{k}' g^{(h)}_{\omega_1,-\omega_1}(\vec{k}') g^{(h')}_{\omega_2,\omega_2'}(\vec{k}' + (2m + \omega_1' - \omega_2) p_F)| \le C \sum_{h=h^*}^{1} [|\sigma_h \gamma^{-h}|^2 + |\sigma_h \gamma^{-h}|] \le C'. \tag{75}$$

Considering $c_n^{(1)}$ (70), it is easy to see that the integrals in the first four terms of (70) are bounded by constants, as there is at least a non diagonal propagator or an ultraviolet one; for the fifth integral the bound is

$$\sum_{h=h^*}^{0} | \int d\vec{k}' \frac{1}{Z_h} \tilde{g}^{(h')}_{\omega,\omega}(\vec{k}') \sum_{h'=h^*}^{0} \frac{1}{Z_{h'}} \tilde{g}^{(h')}_{\omega,\omega}(\vec{k}' + 2n\vec{p}_F)| \le C_1 \sum_{h=h^*}^{1} \frac{1}{(Z_h)^2} \le \frac{2C_1}{c_1 \beta_2 U^2}. \tag{76}$$

A similar bound is found also for the last integral of (70).

Then from the above bounds, if $\frac{\lambda^2}{U^2}$ is small enough one has

$$\frac{\lambda^2}{1 + \lambda^2 c_n^{(1)}} \le 2\lambda^2. \tag{77}$$

We find a solution of (72) as the fixed point $\bar{\Phi}$ of $\mathbf{T}_\lambda$ in $\mathcal{B}$. Note that $\mathcal{B}$ is invariant under the action of $\mathbf{T}_\lambda(\Phi)$, as by (75)

$$|\mathbf{T}_{\lambda,n}(\Phi)| \le 2C\lambda^2 [\sup_{k=\pm 1,\pm 2} \frac{|\lambda|^{\frac{|n+k|}{10}}}{|\lambda|^{\frac{|n|}{10}}}] |\lambda|^{\frac{|n|}{10}} |\sigma| \le \frac{1}{2} |\lambda|^{\frac{|n|}{10}} |\sigma|. \tag{78}$$

so that for λ small enough

$$||\mathbf{T}_\lambda(\Phi)|| \le 1 \tag{79}$$

Moreover, by a similar argument

$$\|\mathbf{T}_\lambda(\Phi) - \mathbf{T}_\lambda(\Phi')\|_{\mathcal{F}} \le \frac{1}{2} \|\Phi - \Phi'\|_{\mathcal{F}}, \tag{80}$$

so that $\mathbf{T}_\lambda$ is a contraction. It is also evident that $\mathbf{T}_\lambda(0) \in \mathcal{B}$. Hence, by the contraction mapping principle, there is a unique fixed point $\bar{\Phi}$ of $\mathbf{T}_\lambda$ in $\mathcal{B}$, which can be obtained as the limit of the sequence $\Phi^{(k)}$ defined through the recurrence equation $\Phi^{(k+1)} = \mathbf{T}_\lambda(\Phi^{(k)})$, with any initial condition $\Phi^{(0)} \in \mathcal{B}$. If we choose $\Phi^{(0)} = 0$, we get

$$\|\bar{\Phi}\|_{\mathcal{F}} \le \sum_{i=1}^{\infty} \|\Phi^{(i)} - \Phi^{(i-1)}\|_{\mathcal{F}} \le \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} \|\Phi^{(1)}\|_{\mathcal{F}} \le \|\Phi^{(1)}\|_{\mathcal{F}}, \tag{81}$$

which immediately imply that $\|\bar{\Phi}\|_{\mathcal{F}} \le 1$. Then if $\frac{U^2}{\lambda^2}$ is large enough there is a solution Φ of (69) which is a function of λ, U, σ; in the following section we insert it in (6) with $n = 1$ looking for a solution σ which is function of λ, U.

B. Determination of $\lambda\hat{\varphi}_1$

We have now to insert $\Phi(\lambda, U, \sigma)$ found above in (6) with $n = 1$, in order to find σ in a self consistent way. We can rewrite (6) as

$$\hat{\varphi}_1 = -\lambda^2 c_1^{(1)}(\sigma) \hat{\varphi}_1 + \lambda \hat{\rho}_1^{(1)}(\lambda, U, \sigma, \Phi), \tag{82}$$

where

$$\begin{aligned} c_1^{(1)}(\sigma) = & \int d\vec{k}' \Big\{ \sum_{h=h^*}^{0} \frac{\tilde{g}^{(h)}_{-1,1}(\vec{k}')}{\sigma} + \tilde{g}^{(1)}_{1,1}(\vec{k}')\, \tilde{g}^{(1)}_{1,1}(\vec{k}' + 2\vec{p}_F) \\ & + \sum_{\omega=\pm 1} \Big[\tilde{g}^{(1)}_{1,1}(\vec{k}') \sum_{h=h^*}^{0} \frac{1}{Z_h} \tilde{g}^{(h)}_{\omega,\omega}(\vec{k}' + (3-\omega)\vec{p}_F) + \\ & \sum_{h=h^*}^{0} \frac{1}{Z_h} \tilde{g}^{(h)}_{\omega,\omega}(\vec{k}')\, \tilde{g}^{(1)}_{1,1}(\vec{k}' + (1+\omega)\vec{p}_F) \Big] \Big\} \\ & = -F(\sigma, \lambda, U) + \tilde{c}_1(\sigma) \end{aligned} \tag{83}$$

and more explicitly

$$F(\sigma,\lambda,U) \equiv \sum_{h=h^*}^{0} \int d\vec{k}' \frac{\tilde{g}^{(h)}_{-1,1}(\vec{k}')}{\sigma} = \tag{84}$$

$$\sum_{h=h^*}^{0} \int d\vec{k} \frac{1}{Z_h} \frac{\sigma_h}{\sigma} \frac{f_h(\vec{k})}{k_0^2 + \sin^2 k' + (1-\cos k')^2 + \sigma_h^2},$$

where σ_h, Z_h verify (57), and $|\tilde{c}_1| \le C$ where C is a constant.

Then by (84) and (68) we obtain

$$\frac{\lambda^2}{\eta_1}[(\frac{|\sigma|}{A})^{-\eta_2} - 1][a^{-1} + U f_1(\lambda, U, \sigma)] + \lambda\sigma f_2(\lambda, U, \sigma) = 1, \tag{85}$$

with $|f_1|, |f_2| \le C$ and $\eta_1 = \beta_1 U + \tilde{\eta}_1$, $\eta_2 = \beta_1 U + \tilde{\eta}_2$, $|\tilde{\eta}_1|, |\tilde{\eta}_2| \le CU^2$, and C, a, β_1, A are positive constants. (85) is a non-BCS or *anomalous* self consistence equation describing a superconductor whose normal state is a Luttinger liquid; if $U = 0$ it reduces to a standard BCS equation. If the electron-electron interaction is larger than the electron-phonon interaction,*i.e.* if $\frac{|\lambda|}{|U|}$ is small, thus the result is very sensitive to the sign of U.

a) In the *attractive* case $U < 0$ there is *no solution* $|\sigma| < 1$ if $\frac{\lambda^2}{U^2}$ is small enough. In fact all the terms in the l.h.s. of (85) can be taken arbitrary small if U, λ and $\frac{\lambda^2}{|U|}$ are small enough so there is no $|\sigma| \le F_0$ solving (85). Note that this is also true considering (5), *i.e.* there is no solution $\varphi \in \Omega$ of (5) if $\frac{\lambda^2}{U^2}$ is small enough for any $\varphi \in \Omega$.

b) In the *repulsive* case $U > 0$ for $\frac{\lambda^2}{|U|^2}$ small enough there is a solution of the form (7), as $\beta, L \to \infty$. Note that this expression for the gap is very different with respect to the BCS like form of the $U = 0$ case; in particular it is as large as $Ae^{-\frac{a}{\lambda^2}|\frac{\lambda^2}{a\eta}||\log||\frac{\lambda^2}{a\eta}|} >> Ae^{\frac{-a}{\lambda^2}}$. A similar expression for the gap appears in the interacting Kondo problem, see Ref. [22] and in superconductors whose ground state is a Luttinger liquid, see Ref. [25]. It is easy to check that corresponding to such solution the Hessian is positive definite.

Finally I discuss shortly how to solve the full (5) without truncating the r.h.s. at the second order. At finite L we can use a contraction method to find a fixed point of (5). However the fact that for the higher order terms we have only bounds, and we cannot make use their exact expressions like was done for (6), has the effect that we can only get a weaker bound for the decay of the solution *i.e.* a power law decay $|\lambda\hat{\varphi}_n| \le \frac{C_N\sigma}{|n|^N}$ instead than an exponential one. This was also what we could get in Ref. [6] in the commensurate case. In the incommensurate case this forbids to take the $L \to \infty$ limit, just because we have proved in Ref. [24] the convergence of $\hat{\rho}_n$ in that limit only if $\lambda\hat{\varphi}_n$ have an exponential decay; on the contrary in the commensurate case convergence holds with a weak decay condition and the limit can be taken. This problem seems however merely technical, and it could be solved proving convergence under power law decay (see discussion at the end of §2.B); this would allow to prove Peierls instability in the incommensurate case without truncating the density expansion.

IV. CONCLUSIONS

In the perturbative regime of small U there is Peierls instability then for $U \ge U_{c,1}$ and for $U \le U_{c,2}$ there is not. It is an interesting open question whether $U_{c,1} = U_{c,2} = U_c$ or not, and if they are equal weather U_c is greater, smaller or equal to zero. We are at the moment not able to answer to this question; we can only say that when $U = 0$ or U small with respect to λ^2 the iterative method for solving (6) is not working. Let us consider in fact the $U = 0$ case. In that case (85) with $n = 1$ is a BCS like equation and gives $|\sigma| = Ae^{\frac{-[a+g]}{\lambda^2}}$, where $|g(U,\lambda)| \le C|\lambda|$. However we are not able to find a solution to the equation for Φ; the contraction method used in the large $\frac{U}{\lambda}$ case here fails as there are integers n such that $2np_F$ is very close to $2p_F$ modulo 2π (of course analogous considerations can be done if is close to $-2p_F$) *i.e.* $||2np_F - 2\omega p_F||_T$ can very small. For such n, $c_n^{(1)}$ (70) computed at $U = 0$ (*i.e.* $Z_h = 1, \sigma_h = \sigma$) can be written as

$$c_n^{(1)} = c_1^{(1)} + b + O(\frac{||2np_F - 2p_F||_T}{\sigma}) + O(\lambda), \tag{86}$$

where b is a constant. As $1 + \lambda^2 c_1 = O(\lambda^2)$ we have that

$$\frac{\lambda^2}{1 + \lambda^2 c_n} = O(1)$$

instead $O(\lambda^2)$ as it is in the preceding section; so it is not clear how a contraction method could be applied (an explicit computation shows that $\frac{\lambda^2}{1+\lambda^2 c_n}$ times the coefficient of $\hat{\varphi}_{n+2}$ tends to 1 when $\alpha_n \to 0$). This fact could be not simply a technical problem linked to the method. The idea underling the analysis of the preceding section, which is essentially the Peierls idea, is that the process involving the exchange of $2p_F$ is the dominant one, and the others are corrections. However, in the incommensurate case for very large n, $2p_F$ or $2np_F$ are almost the same, due to Umklapp scattering, so it is not clear physically why the first harmonic is the dominant one. Technically this means that there are large n for which $\hat{\rho}_n \simeq \hat{\rho}_1$ (see (86) in which b is negligible with respect to c_1 which is log-diverging) so that the self-consistence equation $\hat{\varphi}_n = \lambda \hat{\rho}_n$ for such n or for $n = 1$ seems similar, and it is not clear why $|\hat{\varphi}_n| << |\hat{\varphi}_1|$, as it should be in order φ_x to be analytic. This problem is absent in the *commensurate* case; in that case, see Ref. [6], it is possible to see that $|c_n^{(1)}| \leq C \log Q$ so that for $Q \leq e^{-\frac{1}{|\lambda|}}$ then $\frac{\lambda^2}{1+\lambda^2 c_n} \leq |\lambda|$ and the contraction method will work, so that Peierls instability is proved.

The mechanism why the above difficulty is avoided in the Holstein-Hubbard model with $U > 0$ and $\frac{U}{\lambda^2}$ large, so that Peierls instability is proved in the incommensurate case, is that the harmonic with $n = 1$ has a non trivial flow, and becomes larger, while $\hat{\varphi}_n$ for $|n| \geq 2$ has no flow; this is in a sense a consequence of the diophantine condition.

The difficulty in finding a solution to (6) when $U = 0$ could mean that there is Peierls instability only for $U > U_c$ with $U_c > 0$, but of course a deeper analysis is necessary to conclude. Note that in the $d = \infty$ limit of the Holstein-Hubbard model an incommensurate CDW is found, see Ref. [19], only for $U > U_c$ with U_c non vanishing positive. On the other hand the existence of Peierls instability when $U = 0$ is usually supported by the analogy of the Holstein model with the *Frenkel-Kontorova* chain in which the extremality condition for the energy gives a dynamical system known as *standard map*. For this model the existence of an incommensurate phonon field minimizing the energy is an application of *KAM* theorem in the perturbative regime (and in the strong coupling regime of the so-called Aubry-Mather theorem, in which φ_x is not smooth but it has infinite many discontinuities).

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